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# Estimation of the Scale Parameter of the Half Logistic Distribution by Blom's Method

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ABSTRACT. Blom's method is used to obtain NBLUE of the scale parameter of the half logistic distribution when the location parameter is known ($\mu$ = 0), based on complete sample. A comparison is made with estimators obtained by four methods: the least square method of obtaining best linear unbiased estimators (BLUE's), the method of maximum likelihood estimators (MLE's), the method of approximate maximum likelihood estimators (AMLE's) and (BLUE's) based on two optimally selected order statistics. An illustrative example using life time data is presented for the distribution.

KEYWORDS: Half logistic distribution, Blom estimation method, location and scale parameters, order statistics.

## 1. Introduction

In 1956 and 1958 Blom[1,2] developed a simplified method of estimation of the location and scale parameters of any arbitrary distribution. These estimators are called "nearly best linear unbiased estimators (NBLUEs). They require only the exact values of the means of order statistic and use the asymptotic approximations for the variances and covariance of order statistics. Blom[2] applied his method for six distributions; rectangular, normal, triangular, right triangular, exponential, and extreme-value distribution. Several applications of Blom method have been given in the literature starting with Sarhan and Greenberg 1962[3]. They summarized the method and applied it, in details, for the extreme value distribution. In 1987, Ragab and Green[4] estimated the parameters of the logistic distribution. In 1991, Balakrishnan and Cohen[5] summarized Blom method and estimated the parameters of the normal distribution. In 1995, Jam-

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joom[6] estimated the parameters of second kind Pareto distribution, and in 2000, Al-Ghufaily[7] estimated one shape parameter of Burr type XII distribution. All results seem to warrant the conclusion that Blom approximative method is highly efficient, easy to apply and can be used for both complete and censored samples. The simplicity of this method came from the fact that the means (but not the covariances) of the ordered variables are known only.

**In section 2** below, we presented Balakrishnan and Cohen[5] summary of Blom's for complete and censored samples with some details of Blom[2] for the special case when a single parameter is unknown.

**Section 3** presents application of Blom method to obtain NBLUE of the scale parameter of the half-logistic distribution when the location parameter is $\mu$ is known ($\mu = 0$), and supported by a numerical example. Comparison of the results is made with four different methods is given in Table 2.

## 2. Basic Formulae of NBLUEs

Let $x_1, x_2, ..., x_n$ be a random sample of size $n$ from any population with p.d.f. $f(x)$ c.d.f. $F(x)$, and let $x_{1:n} \le x_{2:n} \le ..... \le x_{n:n}$ be the order statistics obtained from this random sample. Let

$$Z_i = \frac{X_i - \mu}{\sigma} \qquad 1 \le i \le n$$

be the standardized population random variable. Clearly, the distribution of $Z$ is free of the parameters ($\mu$ and $\sigma$) and hence, the means $a_{i:n}$ , $i = 1, 2, 3, ..., n$ and the covariances of the order statistics $Z_{i:n}$ , $i = 1, 2, 3, ..., n$, from the $Z$ population, are free of them as well. Moreover, in view of the David-Johnson approximation (see Blom[2]), it can be shown that

$$\operatorname{cov}\left(Z_{i:n}, Z_{j:n}\right) = \frac{a_i b_j}{(n+2)A_i A_j} + O\left(\frac{1}{n^2}\right), \qquad 1 \le i \le j \le n \tag{2.1}$$

where

$$\left.\begin{aligned} a_i &= \frac{i}{n+1} \\ b_i &= 1 - a_i \end{aligned}\right\} \tag{2.2}$$

$$A_i = f\left(F^{-1}(a_1)\right) \quad ,\text{for} \qquad i = 1, 2, ..., n \tag{2.3}$$

Blom defined what he called the weighted differences between consecutive transformed Beta Variables.

$$Z^*_{i:n} = A_{i+1}\, Z_{i+1:n} - A_i\, Z_{i:n} \quad , \quad 0 \le i \le n \tag{2.4}$$

where the weights $A_i$ defined in (2.3) are such that $A_0 = A_{n+1} = 0$.

Then

$$E\left(Z^{*}_{i:n}\right) = A_{i+1}\ \alpha_{i+1:n} - A_i\ \alpha_{i:n} \tag{2.5}$$

where

$$\alpha_{i:n} = E(Z_{i:n}) \tag{2.6}$$

$$\operatorname{var}\left(Z^{*}_{i:n}\right) \cong \frac{n}{(n+1)^2(n+2)} \tag{2.7}$$

and

$$\operatorname{cov}\left(Z^{*}_{i:n}, Z^{*}_{j:n}\right) \cong \frac{-1}{(n+1)^2(n+2)} \quad , \quad 0 \le i < j \le n \tag{2.8}$$

independently of the parent distribution $F$.

Suppose that the parameter to be estimated is in the general form

$$\delta = L_1\ \mu + L_2\ \sigma \tag{2.9}$$

Where $L_1$, $L_2$ are two given quantities. For $L_1 = 1$, $L_2 = 0$, and $L_1 = 0$, $L_2 = 1$, the parameter is specialized to $\mu$ and $\sigma$ respectively.

The location parameter $\mu$ and the scale parameter $\sigma$ are to be estimated by means of linear expressions of the form

$$\delta_* = \sum_{i=1}^{n} Q_i X_{i:n} = \sum_{i=1}^{n} Q_i(\mu + \sigma\ Z_{i:n}) \tag{2.10}$$

From (2.4) we can write

$$A_i Z_{i:n} = \sum_{j=0}^{i-1} Z^{*}_{j:n} \quad , \quad 1 \le i \le n \tag{2.11}$$

By writing the coefficients $Q_i$ in (2.10) as

$$Q_i = A_i\ (R_i - R_{i-1}) \quad , \quad 1 \le i \le \mathrm{n} \tag{2.12}$$

and by using (2.11), the linear expression can be written as

$$\begin{aligned} \delta_* &= \sum_{i=1}^{n} A_i(R_i - R_{i-1})\ (\mu + \sigma\ Z_{i:n}) \\ &= \mu \sum_{i=1}^{n} A_i(R_i - R_{i-1}) + \sigma \sum_{i=1}^{n} (R_i - R_{i-1}) \sum_{j=0}^{i-1} Z^{*}_{j:n} \\ &= \sum_{i=0}^{n} R_i \left\{ \mu(A_i - A_{i+1}) - \sigma\ Z^{*}_{i:n} \right\} \end{aligned} \tag{2.13}$$

From (2.13) we get the expected value $\delta_*$ by using (2.5) to be

$$\begin{aligned} E(\delta_*) &= \mu\sum_{i=0}^{n} R_i(A_i - A_{i+1}) - \sigma\sum_{i=0}^{n} R_i(A_{i+1}\alpha_{i+1:n} - A_i\alpha_{i:n}) \\ &= \mu\sum_{i=0}^{n} R_i S_{1i} + \sigma\sum_{i=0}^{n} R_i S_{2i} \end{aligned} \qquad (2.14)$$

where

$$S_{1i} = A_i - A_{i+1} \qquad (2.15)$$

$$S_{2i} = A_i\alpha_{i:n} - A_{i+1}\,\alpha_{i+1:n} \qquad (2.16)$$

Furthermore, from (2.13) we obtain the variance of $\delta_*$ by using (2.7) and (2.8) to be

$$\begin{aligned} \mathrm{Var}\,(\delta_*) &= \sigma^2\left[\sum_{i=0}^{n} R_i^2\ \mathrm{Var}\,(Z_{i:n}^*) + \sum_{\substack{i=0 \\ i\neq j}}^{n}\sum_{j=0}^{n} R_i R_j\ \mathrm{Cov}\,(Z_{i:n}^*, Z_{j:n}^*)\right] \\ \mathrm{Var}\,(\delta_*) &\cong \frac{\sigma^2}{(n+1)^2(n+2)}\left[n\sum_{i=0}^{n} R_i^2 - \sum_{i=0}^{n}\sum_{j=0}^{n} R_i R_j\right] \\ &= \frac{\sigma^2}{(n+1)(n+2)}\sum_{i=0}^{n}(R_i - \overline{R})^2 \end{aligned} \qquad (2.17)$$

Where

$$\overline{R} = \sum_{i=0}^{n}\frac{R_i}{(n+1)} \qquad (2.18)$$

The variance of $\delta_*$ in (2.17) is proportional to:

$$Z = \sum_{i=0}^{n}(R_i - \overline{R})^2\ ; \qquad (2.19)$$

That is minimizing $Z$ is equivalent to minimizing the variance of the estimator $\delta_*$. Now $Z$ in (2.19) is minimized with respect to the $R_i$'s subject, to the two side conditions (constraints)

$$\sum_{i=0}^{n} R_i S_{1i} = L_1 \quad , \quad \sum_{i=0}^{n} R_i S_{2i} = L_2 \qquad (2.20)$$

or they can be put together as

$$\sum_{i=0}^{n} R_i\, S_{ri} = L_r \quad , \quad r = 1,2 \tag{2.21}$$

Using the method of Lagrange multiplier[8]. The two constraints (2.20) should be written as:

$$g_1(R) = \sum_{i=0}^{n} R_i\, S_{1i} - L_1 = 0 \tag{2.22}$$

and

$$g_2(R) = \sum_{i=0}^{n} R_i\, S_{2i} - L_2 = 0 \tag{2.23}$$

If $Z$ in (2.19) has a minimum subject to these constraints (2.20), then the following condition must be satisfied for some real numbers $\lambda_1$ and $\lambda_2$.

$$\frac{dZ}{dR} = \lambda_1 \frac{dg_1(R)}{dR} + \lambda_2 \frac{dg_2(R)}{dR} \tag{2.24}$$

where the numbers $\lambda_1$ and $\lambda_2$ are called Lagrange multipliers. Differentiating (2.19) and the conditions (2.22), (2.23) with respect to $R_i$ and substituting the results in (2.24) we get the solution of the minimum value problem in terms of two Lagrange multipliers as:

$$2\sum_{i=0}^{n} (R_i - \bar{R}) = \lambda_1 \sum S_{1i} + \lambda_2 \sum S_{2i} \tag{2.25}$$

which can be written as

$$R_i - \bar{R} = \lambda_1 S_{1i} + \lambda_2 S_{2i}\,, \tag{2.26}$$

The determination of $\lambda_1$ and $\lambda_2$ is made in the traditional way. Multiplying (2.26) first by $S_{1i}$ and adding from 0 to *n,* second by $S_{2i}$ and again adding from 0 to *n*, taking into consideration that

$$\sum_{i=0}^{n} S_{ri} = 0 \qquad r = 1,2 \tag{2.27}$$

and the two constraints (2.22), (2.23) we get the system equations

$$\left.\begin{aligned} \lambda_1 T_{11} + \lambda_2 T_{12} &= L_1 \\ \lambda_1 T_{21} + \lambda_2 T_{22} &= L_1 \end{aligned}\right\} \tag{2.28}$$

where $T_{11} = \sum S_{1i}^2 \quad , \quad T_{12} = T_{21} = \sum S_{1i}\, S_{2i}$

and $T_{22} = \sum S_{2i}^2$ (2.29)

This system (2.28) can be put in matrix from

$$\underline{T}\ \underline{A} = \underline{L} \tag{2.30}$$

where

$$(T_{rs})_{2\times 2} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

The solution of the system (2.30) gives the Lagragian multipliers $\lambda_1$ and $\lambda_2$ given by

$$\lambda_1 = \frac{1}{T_{11}\ T_{22} - T_{12}^2}\left(T_{22}\ L_1 - T_{12}\ L_2\right) \tag{2.31}$$

and

$$\lambda_2 = \frac{1}{T_{11}\ T_{22} - T_{12}^2}\left(T_{11}\ L_2 - T_{12}\ L_1\right) \tag{2.32}$$

here

$$T_{ij} = \sum_{k=0}^{n} S_{iK}\ \ S_{jK} \quad , \quad 1 \le i \le j \le 2 \quad i, j \in \{1,2\} \tag{2.33}$$

With $R_{is}$ as determined in (2.26), $\delta_*$ in (2.16) is the nearly best linear unbiased estimator of $\delta = L_1, \mu + L_2\ \sigma$

From (2.10) we then have

$$\begin{aligned} \delta_* &= \sum_{i=1}^{n} A_i\left(R_i - R_{i-1}\right) X_{i:n} \\ \delta_* &= \sum_{i=1}^{n} A_i\left\{\lambda_1 (S_{1i} - S_{1,i-1}) + \lambda_2 (S_{2i} - S_{2,i-1})\right\} X_{i:n} \end{aligned} \tag{2.34}$$

where $\lambda_1$ and $\lambda_2$ are as given in (2.31) and (2.32) respectively.

In particular, by setting $L_1 = 1$, $L_2 = 0$ and $L_1 = 0$, $L_2 = 1$, we obtain the unbiased nearly best linear estimators of $\mu$ and $\sigma$ as

$$\mu^* = \sum_{i=1}^{n} Q_{1i}\ X_{i:n} \tag{2.35}$$

and

$$\sigma^* = \sum_{i=1}^{n} Q_{2i}\ X_{i:n} \tag{2.36}$$

Where the coefficients $Q_{1i}$ and $Q_{2i}$ are given by

$$Q_{1i} = \frac{A_i}{T_{11}\ T_{22} - T_{12}^2}\left(T_{22}(S_{1i} - S_{1,i-1}) - T_{12}(S_{2i} - S_{2,i-1})\right) \tag{2.37}$$

$$Q_{2i} = \frac{A_i}{T_{11}\ T_{22} - T_{12}^2}\left(T_{11}(S_{2i} - S_{2,i-1}) - T_{12}(S_{1i} - S_{1,i-1})\right) \tag{2.38}$$

Moreover, we obtain the variances and covariances of the estimators $\mu_*$ and $\sigma_*$ from (2.17) to be

$$\begin{aligned}\text{var}\,(\mu^*) &\cong \frac{\sigma^2}{(n+1)(n+2)(T_{11}\ T_{22} - T_{22}^2)^2}\sum_{i=0}^{n}(T_{22}\ S_{1i} - T_{12}S_{2i})^2 \\ &\cong \frac{\sigma^2}{(n+1)(n+2)(T_{11}\ T_{22} - T_{12}^2)^2}(T_{22}^2\ T_{11} - T_{12}^2\ T_{22}) \\ &\cong \frac{\sigma^2\ T_{22}}{(n+1)(n+2)(T_{11}\ T_{22} - T_{12}^2)} \end{aligned} \tag{2.39}$$

$$\begin{aligned}\text{var}\,(\sigma^*) &\cong \frac{\sigma^2}{(n+1)(n+2)(T_{11}\ T_{22} - T_{12}^2)^2}\sum_{i=0}^{n}(T_{11}\ S_{2i} - T_{12}S_{1i})^2 \\ &\cong \frac{\sigma^2}{(n+1)(n+2)(T_{11}\ T_{22} - T_{12}^2)^2}(T_{11}^2\ T_{22} - T_{12}^2\ T_{11}) \\ &\cong \frac{\sigma^2\ T_{11}}{(n+1)(n+2)(T_{11}\ T_{22} - T_{12}^2)} \end{aligned} \tag{2.40}$$

and

$$\text{Cov}\,(\mu^*, \sigma^*) \cong \frac{\sigma^2 T_{12}}{(n+1)(n+2)(T_{11}\ T_{22} - T_{22}^2)} \tag{2.41}$$

In order to find numerical values of the coefficients in Blom method of estimation, it is necessary to follow the following steps as Blom[2] put it.

Step (1) Calculate the means of the standardized ordered variable $\alpha_{i:n}$ = $E$ $(Z_{i:n})$

Step (2) Compute the weights $A_i$ in (2.3)

Step (3) Compute $A_i\, E(Z_i)$

Step (4) Compute $S_{1i}$ *and* $S_{2i}$ according to (2.15) and (2.16)

Step (5) Compute the elements $T_{rs}$ of $T$ defined by (2.29)

Step (6) Insert the resulting numerical values in (2.37) and (2.38), then substitute in (2.35) and (2.36) to get the unbiased nearly best estimate $\mu^*$ and $\sigma^*$

Step (7) If the variances of $\mu^*$ and $\sigma^*$ are required, the approximate expressions (2.39) and (2.40) are then used

As Blom pointed out that the method could be used both when the frequency function is continuous and when it is discrete Blom (1958)[2]. It is also used to construct unbiased nearly best estimates even when only one parameter $\mu$ and $\sigma$ is unknown, and can easily be adapted to censored data.

### *2.a Censored Samples*

When the samples are Type II censored with $r$ smallest and $s$ largest observations are missing, the estimation of $\mu$ and $\sigma$ should then be based upon the observations $Z_{r+1}, Z_{r+2}, \dots, Z_{n-s}$.

The formulae of $\mu^*$ and $\sigma^*$ in (2.35) and (2.36) and their variances and covariance in (2.39) through (2.41) continue to hold with $S_{1i}$ and $S_{2i}$ replaced by $S^*_{1i}$ and $S^*_{2i}$ respectively, where

$$S^*_{1i} = \begin{cases} -\dfrac{1}{r+1} A_{r+1} & , \quad 0 \le i \le r \\ S_{1i} = A_i - A_{i+1} & , \quad r+1 \le i \le n-s-1 \\ \dfrac{1}{(s+1)} A_{n-s} & \quad n-s \le i \le n \end{cases} \tag{2.42}$$

and

$$S^*_{2i} = \begin{cases} -\dfrac{1}{r+1} A_{r+1}\ \alpha_{r+1:n} & \quad 0 \le i \le r \\ S_{2i} = A_i\ \ \alpha_{i:n} - A_{i+1}\ \ \alpha_{i+1:n} & , \quad r+1 \le i \le n-s-1 \\ \dfrac{1}{(s+1)} A_{n-s}\ \ \alpha_{n-s:n} & \quad n-s \le i \le n \end{cases} \tag{2.43}$$

In this case the unbiased nearly best estimator $\delta_*$ in (2.34) becomes

$$\delta_* = \sum_{i=r+1}^{n-s} A_i \{\lambda_1 (S^*_{1i} - S^*_{1,i-1}) + \lambda_2 (S^*_{2i} - S^*_{2,i-1})\} X_{i:n} \tag{2.44}$$

where $\lambda_1$ and $\lambda_2$ are as given in (2.31) and (2.32), respectively. Then the estimators $\mu^*$ and $\sigma^*$ in (2.35), (2.36) become

$$\mu^* = \sum_{i=r+1}^{n-s} Q_i\ X_{i:n} \tag{2.45}$$

and

$$\sigma^* = \sum_{i=r+1}^{n-s} Q_{2i} X_{i:n} \tag{2.46}$$

where

$$Q_{1i} = \frac{A_i}{T_{11}\, T_{22} - T_{12}^2} \{T_{22}(S_{1i}^* - S_{1,i-1}^*) - T_{12}(S_{2i}^* - S_{2,i-1}^*)\} \tag{2.47}$$

and

$$Q_{2i} = \frac{A_i}{T_{11}\, T_{22} - T_{12}^2} \{T_{11}(S_{2i}^* - S_{2,i-1}^*) - T_{12}(S_{1i}^* - S_{1,i-1}^*)\} \tag{2.48}$$

***2.b. A single Unknown Parameter***

When only one of the parameters $\mu$ or $\sigma$ is unknown, Blom's method should be modified.

***2.b.1 $\mu$ Unknown, $\sigma$ Known:***

The variance of $\delta_*$ in (2.17) or equivalently $Z$ in (2.19) is minimized with respect to the $R_i$'s subject to the single side condition.

$$\sum_{i=0}^{n} R_i\, S_{1i} = L_1 \tag{2.49}$$

Apply the method of Lagrange multipliers, the solution is found to be

$$R_i - \bar{R} = \lambda_1\, S_{1i} \tag{2.50}$$

where

$$\lambda_1 = \frac{L_1}{T_{11}} \tag{2.51}$$

is a Lagrange multiplier defined earlier. Further, the coefficients of the resulting nearly best estimate $\delta_{*1}$ are given by

$$Q_i = A_i \frac{L_1}{T_{11}} (S_{1i} - S_{1,i-1}) \tag{2.52}$$

The mean of $\delta_{*1}$ in (2.14) is given by

$$\begin{aligned} E(\delta_{*1}) &= L_1\, \mu + \sigma \sum_{i=0}^{n} S_{2i}\, R_i \\ E(\delta_{*1}) &= L_1 \left( \mu + \sigma \frac{T_{12}}{T_{11}} \right) \end{aligned} \tag{2.53}$$

Thus the nearly best linear unbiased estimate $\mu^{*\prime}$ of $\mu$ is then obtained by taking $L_1 = 1$ and subtracting the known term $\sigma \frac{T_{12}}{T_{11}}$ from $\delta_*$ .

That is

$$\mu^{*\prime} = \sum_{i=1}^{n} R_i Z_i - \sigma \frac{T_{12}}{T_{11}} \quad . \tag{2.54}$$

The variance of $\mu^{*\prime}$ is approximately

$$\text{var } \mu^* \cong \frac{\sigma^2}{(n+1)(n+2)T_{11}} \tag{2.55}$$

### *2.b.2. μ Known, σ Unknown*

The variance of $\delta_*$ in (2.17) or equivalently, $Z$ in (2.19), in this case, is minimized with respect to $R_i$'s subject to the single side condition

$$\sum_{i=0}^{n} R_i S_{2i} = L_2 \tag{2.56}$$

The solution is found to be

$$R_i - \bar{R} = \lambda_2 S_{2i} \tag{2.57}$$

where

$$\lambda_2 = \frac{L_2}{T_{22}} \tag{2.58}$$

The coefficients of the nearly best estimate $\delta_{*2}$ are given by

$$Q_i = A_i \frac{L_2}{T_{22}} (S_{2i} - S_{2,i-1}) \tag{2.59}$$

Then the mean of $S_{*2}$ is

$$E(\delta_{*2}) = L_2 \left( \frac{\mu\, T_{12}}{T_{22}} + \sigma \right) \tag{2.60}$$

Thus the nearly best linear unbiased estimate $\sigma^{*\prime}$ is obtained by taking $L_2$ = 1 and subtracting the known term $\frac{\mu\, T_{12}}{T_{22}}$ from $\delta_*$. That is

$$\sigma^{*\prime} = \sum_{i=1}^{n} R_i Z_i - \frac{\mu\, T_{12}}{T_{22}} \tag{2.61}$$

The variance of $\sigma^{*\prime}$ is approximately

$$\text{var } \sigma^{*\prime} = \frac{\sigma^2}{(n+1)(n+2)\, T_{22}} \tag{2.62}$$

By considering the data given by[9], in example (1), we shall demonstrate Blom's method to estimate the scale parameter of half-logistic distribution in the next section.

## 3. Application of Blom's Method for Estimating the Scale Parameter of the Half Logistic Distribution (Location Parameter is Known)

The c.d.f. of the half-logistic distribution is given by

$$\begin{aligned} F(y,\mu,\sigma) &= \frac{1-\exp-\left(\dfrac{y-\mu}{\sigma}\right)}{1+\exp-\left(\dfrac{y-\mu}{\sigma}\right)} \quad , \quad \mu \le y < \infty \\ &= 0 \quad , \quad \text{elsewhere} \end{aligned} \tag{3.1}$$

where $\mu$ and $\sigma$ are the location and scale parameters. The standardized variate $Z = \dfrac{Y-\mu}{\sigma}$ follows the standardized half-logistic distributionwith c.d.f. given by

$$\begin{aligned} G(z) &= \frac{2}{1+e^{-z}} - 1 \quad , \quad z \ge 0 \\ &= 0 \quad , \quad \text{elsewhere} \end{aligned} \tag{3.2}$$

Estimation of the parameters of this distribution has been studied by several authors[9],[10],[11]. Let $Z_{1:n}$ , $Z_{2:n}$ , ..., $Z_{n:n}$ be a random sample of size $n$ obtained from (3.2) and let $Z_1 \le Z_2 \le Z_3 \le \ldots\ldots \le Z_n$ be the order statistics obtained from this random sample.

***Example (1)***

The following data represented failure time, in minutes, for a specific type of electrical insulation that was subjected to continuously increasing voltage stress as given by[9]:

12.3, 21.8, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, 151.9

These data are also listed in column 8 of table (2); Chan 1989[9] has shown that the one parameter half-logistic distribution (3.1) with $\mu = 0$ fits the above data extremely well. In this case we have a complete sample of size $n = 12$ and the location is known ($\mu = 0$). NBLUE of the scale parameter σ is now desired.

For estimating the scale parameter σ by linear function of order statistics, Blom's NBLUE $\sigma^*$ is in the form $\sigma^* = \Sigma Q_i X_{i:n}$, the successive 7-steps mentioned earlier is now followed. For step one, the expected values of the $i^{th}$ order statistics for the standardized half-logistic distribution $\alpha_{i:n} = E_i(Z_{i:n})$, $1 \le i \le n$ can be calculated using the following recurrence relations given by[10]

$$\alpha_{1:n+1} = 2\left(\alpha_{1:n} - \frac{1}{n}\right), \qquad n \ge 1 \tag{3.3}$$

$$\alpha_{2:n+1} = \frac{(n+1)}{n} - \frac{(n-1)}{2}\alpha_{1:n+1} \qquad n \ge 1 \tag{3.4}$$

$$\alpha_{i+1:n+1} = \frac{1}{i}\left[\frac{n+1}{n-i+1} + \frac{n+1}{2}\alpha_{i-1:n} - \frac{n-2i+1}{2}\alpha_{i:n+1}\right], \quad 2 \le i \le n; \tag{3.5}$$

such that $\alpha_{1:1} = E(Z_{1:1})$ ln 4. Expected values of the $i^{th}$ order statistics for $n$ = 12 are given in column (1) of table (1). Next, the weights of $A_i$ in (2.3) are

$$A_i = \frac{(n+1)^2 - i^2}{2(n+1)^2}, \quad 1 \le i \le n, \tag{3.6}$$

$$A_0 = A_{n+1} = 0 \tag{3.7}$$

These equations give column (2) of table (1). For $n$ = 12 we construct the first 3 columns of table (2). For step 4, $S_{2i}$ of (2.16) can be simplified as follows

$$S_{2i} = \begin{cases} \dfrac{-n(n+2)}{2(n+1)^2}\,\alpha_{1:n} & i = 0 \\[2ex] \dfrac{1}{2(n+1)^2}\left\{\left((n+1)^2 - i^2\right)[a_{i:n} - \alpha_{i+1,n}] + (2i+1)\alpha_{i+1,n}\right\}, & i \le i \le n \end{cases} \tag{3.8}$$

where $\alpha_{i:n}$, $\alpha_{i+1:n}$ can be obtained from column (1) of table (1). Values of $S_{2i}$ are listed in column (4) and the coefficients $Q_i$ in (2.59) are given in column (5) and finally, using the data of Example 1 and the coefficients of column (5) the NBLUE of the scale parameter $\sigma$ when the location parameter $\mu$ is known ($\mu$ = 0) is found to be

$$\sigma^{*\prime} = 46.48649317$$

and its variance by (2.62) is

$$\text{var}\left(\sigma^{*\prime}\right) = 0.04487\ (46.48649)$$
$$= 2.08585$$

Balakrishnan and Cohen[5] and Balakrishnan and Chan[11] lobtained $\sigma$ for this example by four methods. Comparison of our results with their results is illustrated in table (2), which shows that the methods gave very close results of $\sigma^*$ and its variance.

TABLE 1. Calculations of nearly best linear unbiased estimator of the scale parameter of the half-logistic distribution, sample size = 12.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $I$ | $E(X_i)$ | $A_i$ | $A_i^* E(X_i)$ | $S_{2i}$ | $Q_i$ | Data | $Q_i^* Q_i$ |
| 0 | 0 | 0 | 0 | – 0.077235201 | 0 | 0 | 0 |
| 1 | 0.15539 | 0.497041 | 0.077235201 | – 0.076024515 | 0.004913988 | 12.3 | 0.060442046 |
| 2 | 0.31395 | 0.488166 | 0.153259716 | – 0.072979442 | 0.01213879 | 21.8 | 0.264625612 |
| 3 | 0.47793 | 0.473373 | 0.226239158 | – 0.067570816 | 0.020907432 | 24.4 | 0.510141351 |
| 4 | 0.64907 | 0.452663 | 0.293809973 | – 0.060443221 | 0.02634682 | 28.6 | 0.753519043 |
| 5 | 0.83151 | 0.426036 | 0.354253194 | – 0.050424755 | 0.034854411 | 43.2 | 1.505710556 |
| 6 | 1.02843 | 0.393491 | 0.404677949 | – 0.038005408 | 0.039906513 | 46.9 | 1.871615479 |
| 7 | 1.24689 | 0.35503 | 0.442683357 | – 0.022090924 | 0.046138962 | 70.7 | 3.262024599 |
| 8 | 1.49613 | 0.310651 | 0.464774281 | – 0.00210472 | 0.050700623 | 75.3 | 3.817756888 |
| 9 | 1.79324 | 0.260355 | 0.466879 | 0.023551984 | 0.05454777 | 95.5 | 5.209312034 |
| 10 | 2.17166 | 0.204142 | 0.443327016 | 0.057780057 | 0.057059114 | 98.1 | 5.597499069 |
| 11 | 2.71489 | 0.142012 | 0.385546959 | 0.10770711 | 0.057899017 | 138.6 | 8.024803697 |
| 12 | 3.75642 | 0.073964 | 0.277839849 | 0.277839849 | 0.102758675 | 151.9 | 15.60904279 |
| 13 | 0 | 0 | 0 | 0 | 0 | | |
| | | | | | | | σ = 46.48649317 |

TABLE 2. Comparison of estimators of the scale parameter of the half-logistic distribution obtained by four methods.

| Method | $\sigma^{*'}$ | Var ($\sigma^{*'}$) |
|---|---|---|
| NBLUE | 46.48649317 | 0.04487 $\sigma$ |
| BLUE'S | 48.01 | 0.05848 $\sigma$ |
| BLUE'S based on 2 optimally selected order statistics | 48.41 | 0.06647 $\sigma$ |
| MLE | 47.41609 | – |
| AMLE | 47.41613 | 0.05801 $\sigma$ |

From the above results, we notice that the methods gave very close results of $\sigma^{*\prime}$ and its variance.

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# تقدير معلمة القياس لتوزيع Half Logistic باستخدام طريقة بلوم

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*المستخلص.* يتناول هذا البحث تقديرا لمعلمة القياس لتوزيع Half-Logistic بافتراض أن معلمة الموضع معلومة ، وذلك باستخدام طريقة بلوم . هذه الطريقة هي إحدى طرق تقدير معلمتي الموضع والقياس باستخدام دوال خطية في الإحصاءات الترتيبية ، ويطلق على المقدر اسم « أقرب وأفضل مقدر خطي غير متحيز » ويرمز له بالرمز (NBLUE). تم دعم النتائج النظرية لهذه الطريقة بمثال عددي لعينة عشوائية كاملة حجمها $(n = 12)$. كما تم مقارنة النتائج بالنتائج المتوفرة في الأبحاث العلمية باستخدام أربعة طرق للتقدير وهي : طريقة الإمكان الأعظم ، طريقة الإمكان الأعظم التقريبية ، طريقة المربعات الصغرى ، وطريقة المربعات الصغرى التي تعتمد على استخدام أفضل إحصاءين ترتيبيين . وقد وجد أن الفرق في قسم معلمة القياس المقدرة بواسطة الطرق الخمسة صغير جدا وامتاز مقدر المربعات الصغرى التقريبي بأقل تباين .